Cutting planes in mixed integer programming

Wei-Kun Chen

School of Mathematics and Statistics
Beijing Institute of Technology

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Agenda

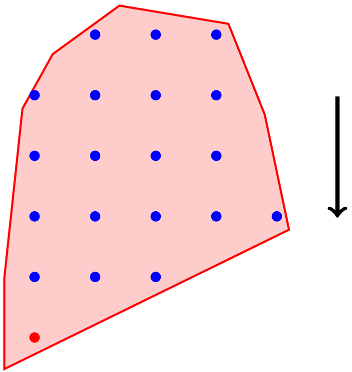
1. Background on cuts
2. Mixing cuts
3. Sequentially lifted cuts

Agenda

Mixed integer programming (MIP)

$$\begin{aligned} \min_{x,y} \quad & c^\top x + g^\top y \\ \text{s.t.} \quad & Ax + By \le d, \\ & (x,y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^m. \end{aligned}$$

- **Applications**: supply chain, electrical power, finance, transportation, work force management ...
- **Algorithm**: branch-and-**cut**.

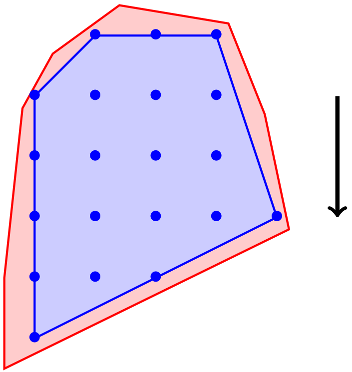


An MIP is indeed an LP

$$\min\left\{c^\top x + g^\top y \,:\, (x,y) \in \mathcal{X}\right\} \Longleftrightarrow$$

$$\min\left\{c^\top x + g^\top y \,:\, (x,y) \in \operatorname{conv}(\mathcal{X})\right\}.$$

- $\mathcal{X} = \{(x,y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^m \,:\, Ax + By \le d\}$.
- $\operatorname{conv}(\mathcal{X}) = \{(x,y) \in \mathbb{R}_+^{n+m} \,:\, \bar{A}x + \bar{B}y \le \bar{d}\}$.



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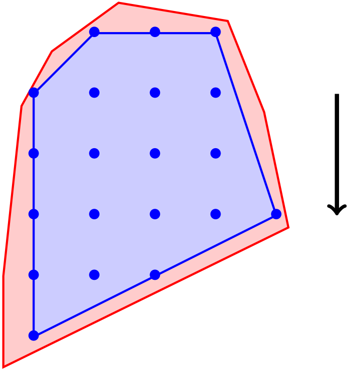

Two difficulties:

- Computing $\operatorname{conv}(\mathcal{X})$ is hard.
- $\bar{A}x + \bar{B}y \le \bar{d}$ is potentially huge.

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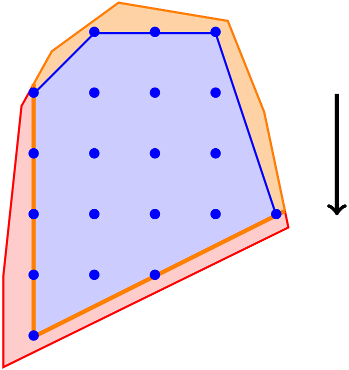

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- Computing $\operatorname{conv}(\mathcal{X})$ is hard.
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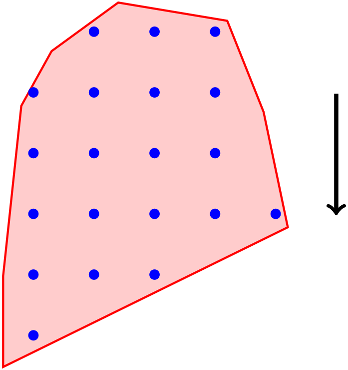
Unnecessary to compute $\operatorname{conv}(\mathcal{X})$.

Cutting plane algorithm

1. Solve the **linear programming (LP) relaxation problem** to obtain its solution (x^*, y^*).
2. (i) If (x^*, y^*) satisfies the **integer** constraint
$$x^* \in \mathbb{Z}^n_+,$$
stop with the optimal solution (x^*, y^*).
(ii) Otherwise, find some **valid inequalities** ($\alpha^\top x + \beta^\top y \le \gamma,\ \forall\ (x,y) \in \mathcal{X}$) violated by (x^*, y^*) (**cuts**). Add these cuts to the problem and solve the LP again.



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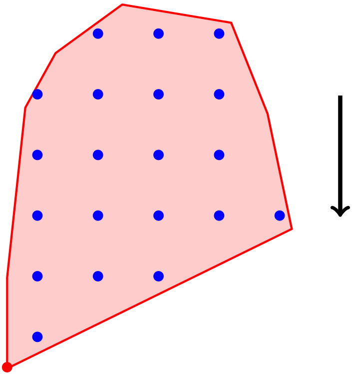

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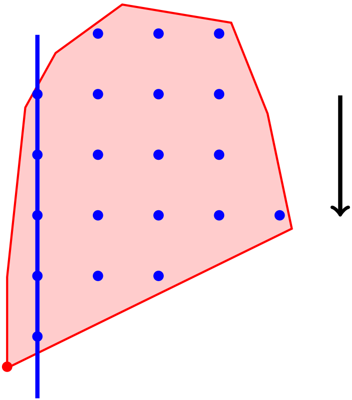

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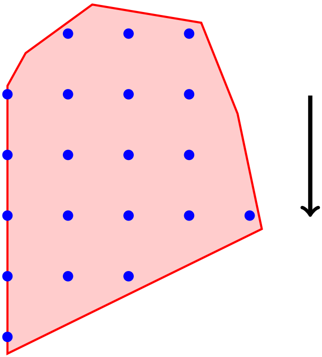

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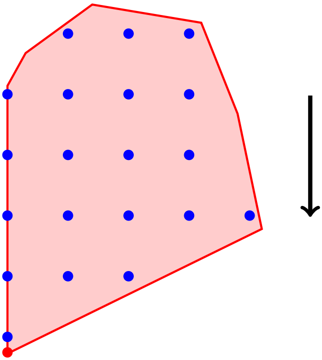

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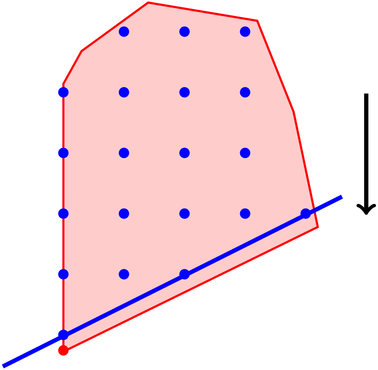

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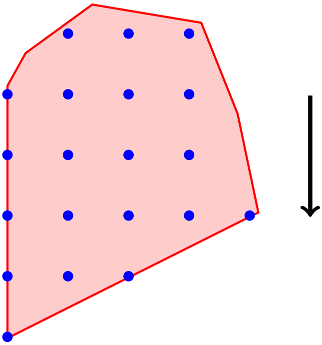

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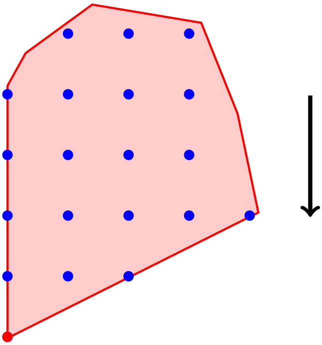

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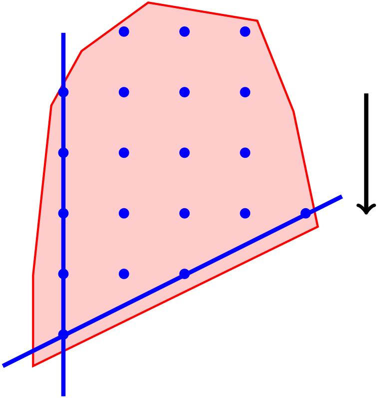


This talks: cuts (computation, separation).

Agenda

1. Background on cuts
2. Mixing cuts
3. Sequentially lifted cuts

An illustrative example

- $y \geq 2x_1, \ y \geq 3x_2, \ y \geq 5x_3, \ x_1, x_2, x_3 \in \{0, 1\}, \ y \in \mathbb{R}$.
- A stronger valid inequality: $y \geq 2x_1 + x_2$.
  1. $x_2 = 0$: implied by $y \geq 2x_1$.
  2. $x_2 = 1$: implied by $y \geq 3x_2 = 3$.
- A more stronger valid inequality: $y \geq 2x_1 + x_2 + 2x_3$.
  1. $x_3 = 0$: implied by $y \geq 2x_1 + x_2$.
  2. $x_3 = 1$: implied by $y \geq 5x_3 = 5$.

Mixing inequalities

- **Mixing set**: $\mathcal{X} = \big\{(x, y) \in \{0,1\}^n \times \mathbb{R} : y \geq a_i x_i,\ i \in [n] := 1, \ldots, n\big\}$.
- **Mixing inequality** [Atamtürk-Nemhauser-Savelsbergh, 2000], [Günlük-Pochet, 2001]

$$y \geq \sum_{\tau=1}^{s} (a_{i_\tau} - a_{i_{\tau-1}}) x_{i_\tau}, \tag{1}$$

where $\mathcal{S} := \{i_1, \ldots, i_s\} \subseteq [n]$ such that $a_{i_1} \leq \cdots \leq a_{i_s}$ $(a_{i_0} = 0)$.

- $\mathcal{S}$ grows exponentially with n but finding the most violated (1) (by (x^*, y^*)) can be done in $\mathcal{O}(n \log n)$.
  - Reorder variables x_i, $i \in [n]$, such that $x_1^* \geq \cdots \geq x_n^*$ and set $\mathcal{S} := \{1\}$.
  - For each $i \in [n] \backslash \{1\}$, set $\mathcal{S} := \mathcal{S} \cup \{i\}$ if $a_i > a_k$, where k is the last index added into $\mathcal{S}$.

Application

Chance-constrained program (CCP) with random RHS [Miller-Wagner, 1965]

$$\begin{aligned} \min \quad & c^\top z \\ \text{s.t.} \quad & \mathbb{P}\{Tz \geq \xi\} \geq 1 - \epsilon, \\ & z \in \mathcal{Z}. \end{aligned} \tag{CCP}$$

- ξ is an m-dimensional nonnegative random vector with **discrete distribution**:

$$\mathbb{P}\{\xi = \xi_i\} = p_i, \ i \in [n].$$

- $\mathcal{Z} = \{z \in \mathbb{Z}_+^p \times \mathbb{R}_+^q \ : \ Az \leq b\}$.

An equivalent MIP formulation for CCP [Ruszczyński, 2002]

We introduce $Tz = v$ and for each i, a binary variable x_i, where $x_i = 1$ guaranteeing $v \geq \xi_i$. Then the CCP can be equivalently formulated as:

$$\begin{aligned} \min \quad & c^\top z \\ \text{s.t.} \quad & \mathbb{P}\{Tz \geq \xi\} \geq 1 - \epsilon, \\ & z \in \mathcal{Z}. \end{aligned} \qquad \text{(CCP)}$$

$$\begin{aligned} \min \quad & c^\top z \\ \text{s.t.} \quad & Tz = v, \; z \in \mathcal{Z}, \\ & v \geq \xi_i x_i, \; \forall \; i \in [n], \\ & \sum_{i=1}^{n} p_i x_i \geq 1 - \epsilon, \\ & v \in \mathbb{R}^m, \; x \in \{0,1\}^n. \end{aligned} \qquad (2)$$

- **Mixing sets**: $\mathcal{X}(j) = \left\{(x, v_j) \in \{0,1\}^n \times \mathbb{R} : v_j \geq \xi_{ij} x_i, \; i \in [n]\right\}$.
- More investigations on mixing sets with constraint $\sum_{i=1}^{n} p_i x_i \geq 1 - \epsilon$ [Luedtke-Ahmed-Nemhauser, 2010], [Küçükyavuz, 2012], [Abdi-Fukasawa, 2016], [Zhao-Huang-Zeng, 2017], [Kılınç-Karzan-Küçükyavuz-Lee, 2022].

A generic implementation

- **Variable bounds relations**:

$$y \star ax + b, \quad \star \in \{\geq, \leq\}. \tag{3}$$

- During the presolve process, MIP solvers detect relations (3) from two-variable constraints or general constraints by *probing* [Savelsbergh, 1994].
- Had already be used to, e.g., tighten bounds of variables through (node) presolve, guide branching, and enhance cuts separation.
- The mixing cuts can be derived from variable bounds relations in which x is a binary variable and y is a non-binary variable.

$$\mathcal{X} = \left\{(x, y) \in \{0,1\}^n \times \mathbb{R} : y \geq a_i x_i,\ i \in [n] := 1, \ldots, n\right\}$$

≥-mixing cuts

$$y \geq a_i x_i + b_i,\ x_i \in \{0,1\},\ i \in \mathcal{N},\ y \in [\ell, u]. \tag{4}$$

Normalization: $0 < a_i \leq u - \ell$ and $b_i = \ell$ for all $i \in [n]$.

(i) If $a_i < 0$, variable x_i can be complemented by $1 - x_i$. If $a_i = 0$, $y \geq a_i x_i + b_i$ can be removed from (4) and $\ell' := \max\{\ell, b_i\}$ is the new lower bound for y ($a_i > 0$).

(ii) If $a_i + b_i \leq \ell$, by $a_i > 0$ (from (i)), constraint $y \geq a_i x_i + b_i$ is implied by $y \geq \ell$ and hence can be removed from (4) ($a_i + b_i > \ell$).

(iii) If $b_i > \ell$, by $a_i > 0$ (from (i)), $\ell' := b_i$ is the new lower bound for y; if $b_i < \ell$, by $a_i + b_i > \ell$ (from (ii)), relation $y \geq a_i x_i + b_i$ can be changed into $y \geq (a_i + b_i - \ell)x_i + \ell$ ($b_i = \ell$).

(iv) If $a_i > u - \ell$, by $b_i = \ell$ (from (iii)), $x_i = 0$ must hold and constraint $y \geq a_i x_i + \ell$ can be removed from (4) ($a_i \leq u - \ell$).

≥-mixing cut:

$$y - \ell \geq \sum_{\tau=1}^{s} (a_{i_\tau} - a_{i_{\tau-1}}) x_{i_\tau},$$

where $\mathcal{S} := \{i_1, \ldots, i_s\} \subseteq \mathcal{N}$ such that $a_{i_1} \leq \cdots \leq a_{i_s}$ $(a_{i_0} = 0)$.

$\leq$-mixing cuts

$$y \leq -a_j x_j + b_j,\ x_j \in \{0,1\},\ j \in \mathcal{M},\ y \in [\ell, u].$$

Normalization: $0 < a_j \leq u - \ell$ and $b_j = u$ for all $j \in \mathcal{M}$.

$\leq$-**mixing cut**:

$$u - y \geq \sum_{\tau=1}^{t} (a_{j_\tau} - a_{j_{\tau-1}}) x_{j_\tau},$$

where $\mathcal{T} := \{j_1, \ldots, j_t\} \subseteq \mathcal{M}$ such that $a_{j_1} \leq \cdots \leq a_{j_t}$ $(a_{j_0} = 0)$.

Conflict inequality

Observation: $y \geq 2x_1$, $y \leq -5x_2 + 6$, $x_1, x_2 \in \{0,1\} \Rightarrow x_1 + x_2 \leq 1$.

$$y \geq a_i x_i + \ell,\ x_i \in \{0,1\},\ i \in \mathcal{N},$$
$$y \leq -a_j x_j + u,\ x_j \in \{0,1\},\ j \in \mathcal{M},\ y \in [\ell, u].$$

Conflict inequality:

$$x_{i'} + x_{j'} \leq 1,$$

where $i' \in \mathcal{N}$ and $j' \in \mathcal{M}$ such that $a_{i'} + a_{j'} > u$.

CCPs instances

Transportation problem [Luedtke-Ahmed-Nemhauser, 2010]

$$\min_{x \in \mathbb{R}_+^{n\times m}} \left\{ \sum_{i\in[n]} \sum_{j\in[m]} c_{ij}x_{ij} : \sum_{j\in[m]} x_{ij} \le M_i,\ i \in [n],\ \mathbb{P}\left\{ \sum_{i\in[n]} x_{ij} \ge d_j, j \in [m] \right\} \ge 1 - \epsilon \right\},$$

40 instances, https://jrluedtke.github.io.

Lot sizing problem [Zhao-Huang-Zeng, 2017]

$$\min_{(x,w)\in\mathbb{R}_+^T\times\{0,1\}^T} \left\{ \sum_{t\in[T]} \left(c_t x_t + f_t w_t + h_t \mathbb{E}\left(\left(\sum_{j\in[t]} x_j - \sum_{j\in[t]} d_j \right)^+ \right) \right) : \right.$$

$$\left. x_t \le M_t w_t,\ t \in [T],\ \mathbb{P}\left\{ \sum_{j\in[t]} x_j \ge \sum_{j\in[t]} d_j,\ t \in [T] \right\} \ge 1 - \epsilon \right\},$$

90 instances, https://sites.pitt.edu/~bzeng.

Computational results: CCPs

- **Hardware**: a cluster of Intel(R) Xeon(R) Gold 6140 CPU @ 2.30GHz computers.
- **MIP solver**: SCIP 8.0.0.
- **Time limit**: 72 00 seconds.

Problems	DEFAULT			NO MIXING CUT		
	Solved	Time	Nodes	Solved	Time	Nodes
Transportation	32	290.8	66	9	3630.5	14245
Lot sizing	72	2003.4	103964	59	2790.2	161084

- The mixing cuts can effectively improve the performance of solving MIP formulations of CCPs.

Computational results: MIPLIB 2017 Benchmark

240 instances, https://miplib.zib.de/index.html.

Bracket	#	DEFAULT			NO MIXING CUT			R_T	R_N
		Solved	Time	Nodes	Solved	Time	Nodes		
[0, 7200]	123	122	220.5	3124	120	230.9	3476	1.05	1.11
[10, 7200]	112	111	323.2	4104	109	340.6	4610	1.05	1.12
[100, 7200]	79	78	817.6	12525	76	871.7	14205	1.07	1.13
[1000, 7200]	40	39	2253.0	50486	37	2441.3	61301	1.08	1.21

- The mixing cuts hold potential for practically solving generic MIPs.
- Problem fhnw-binpack4-48 goes from $7200+$ seconds to 72.2 seconds.

Agenda

Integer knapsack cover relaxation

$$\begin{aligned} \min_{x,y} \quad & c^\top x + g^\top y \\ \text{s.t.} \quad & Ax + By \le d, \\ & (x,y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^m. \end{aligned} \tag{5}$$

Integer knapsack cover (row) relaxation:

$$\mathcal{X} := \left\{ x \in \mathbb{Z}_+^n : a^\top x \ge b \right\},$$

where $a^\top x \ge b$ is a row in (5) with $a_i \in \mathbb{Z}_{++}$ and $b \in \mathbb{Z}_{++}$.

Applications: cutting stock problem, heterogeneous vehicle routing problem, mixed pallet design (MPD) problem.

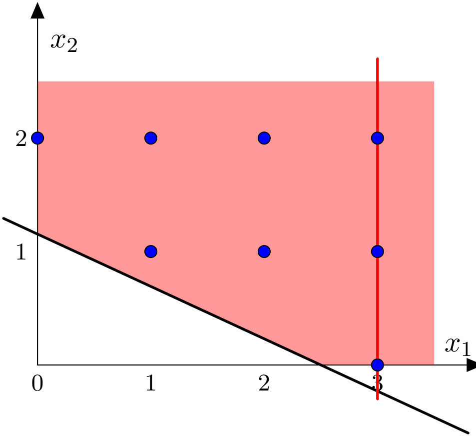
Investigations on $\mathrm{conv}(\mathcal{X})$: [Pochet-Wolsey, 1995], [Mazur, 1999], [Yaman, 2007].

An illustrative example

- $\mathcal{X} = \{x \in \mathbb{Z}_+^2 : 6x_1 + 13x_2 \geq 15\}$
- Fixing $x_2 = 0$, $\mathcal{X}$ reduces to

  $$\mathcal{X}(\{1\}) := \{x_1 \in \mathbb{Z}_+ : 6x_1 \geq 15\}.$$

- $x_1 \geq 3$ is **valid for** $\mathcal{X}(\{1\})$ but invalid for $\mathcal{X}$.
- **Solution**: rotating.



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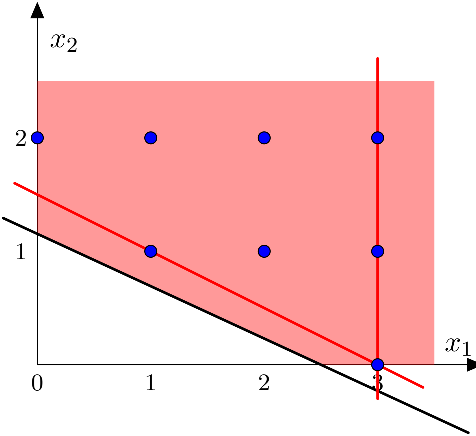

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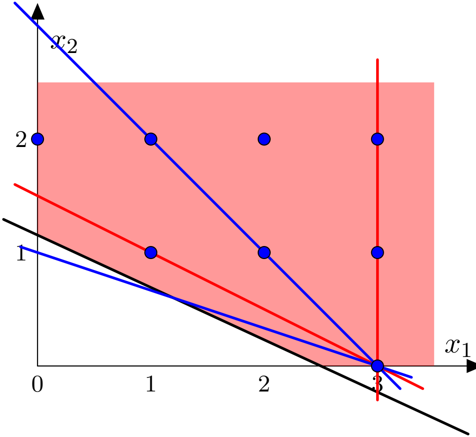
- **Solution**: rotating.
- The other two are too weak or invalid.



An illustrative example cont.

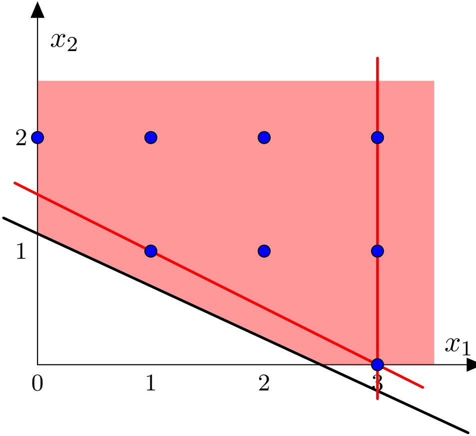
- $\mathcal{X} = \{x \in \mathbb{Z}_+^2 : 6x_1 + 13x_2 \geq 15\}$.

$x_1 + \alpha_2 x_2 \geq 3$ is a valid inequality for $\mathcal{X}$ if and only if

$$\alpha_2 \geq \frac{3 - x_1}{x_2}, \ \forall x \in \mathcal{X}, \ x_2 \geq 1.$$

This is equivalent to

$$\alpha_2 \geq z = \max_{x \in \mathbb{Z}_+^2} \ \frac{3 - x_1}{x_2} \quad \text{s.t.} \quad 6x_1 + 13x_2 \geq 15, \quad x_2 \geq 1.$$



$\alpha_2 = z = 2 \Rightarrow$ A strong valid inequality $x_1 + 2x_2 \geq 3$.

Lifting

- $\mathcal{X} := \{x \in \mathbb{Z}_+^n : a^\top x \geq b\}$.
- Fix $x_i = 0$ for all $i \in [n] \backslash \{j\}$.
- Let $b = ka_j + r$ where $k, r \in \mathbb{Z}_+$ and $1 \leq r \leq a_j$.
- $rx_j \geq r(k+1)$ is a **valid inequality** of $\mathcal{X}(\{j\}) := \{x_j \in \mathbb{Z}_+ : a_j x_j \geq b\}$.
- $rx_j + \alpha_\ell x_\ell \geq r(k+1)$ is valid inequality for $\mathcal{X}(\{j, \ell\})$ **if and only if**

$$\alpha_\ell \geq \frac{r(k+1) - rx_j}{x_\ell}, \ \forall x \in \mathcal{X}(\{j, \ell\}), \ x_\ell \geq 1.$$

Lifting cont.

- Equivalent to

$$\alpha_\ell \geq z_\ell = \max_{x \in \mathbb{Z}_+^2} \quad \frac{r(k+1) - rx_j}{x_\ell}$$
$$\text{s.t.} \quad a_j x_j + a_\ell x_\ell \geq b,$$
$$x_\ell \geq 1.$$

- Setting $\alpha_\ell = z_\ell$, we obtain a valid inequality

$$rx_j + \alpha_\ell x_\ell \geq r(k+1)$$

for $\mathcal{X}(\{j, \ell\})$.

Lifting cont.

- Assume that

$$rx_j + \sum_{i \in \mathcal{S}} \alpha_i x_i \geq r(k+1)$$

is valid for $\mathcal{X}(\mathcal{S} \cup \{j\})$ where $\mathcal{S} \subseteq [n] \backslash \{j\}$.

- For $\ell \in [n] \backslash (\mathcal{S} \cup \{j\})$,

$$rx_j + \alpha_\ell x_\ell + \sum_{i \in \mathcal{S}} \alpha_i x_i \geq r(k+1)$$

is a valid inequality for $\mathcal{X}(\mathcal{S} \cup \{j, \ell\})$ **if and only if**

$$\alpha_\ell \geq \frac{r(k+1) - rx_j - \sum_{i \in \mathcal{S}} \alpha_i x_i}{x_\ell}, \ \forall \ x \in \mathcal{X}(\mathcal{S} \cup \{j, \ell\}), \ x_\ell \geq 1.$$

Lifting cont.

- Equivalent to

$$\alpha_\ell \geq z_\ell = \max_{x \in \mathbb{Z}_+^{|\mathcal{S}|+2}} \quad \frac{r(k+1) - rx_j - \sum_{i\in\mathcal{S}} \alpha_i x_i}{x_\ell}$$

$$\text{s.t.} \quad a_j x_j + a_\ell x_\ell + \sum_{i\in\mathcal{S}} a_i x_i \geq b,$$

$$x_\ell \geq 1.$$

- Setting $\alpha_\ell = z_\ell$, we obtain a valid inequality

$$rx_j + \alpha_\ell x_\ell + \sum_{i\in\mathcal{S}} \alpha_i x_i \geq r(k+1),$$

for $\mathcal{X}(\mathcal{S} \cup \{j, \ell\})$.

Sequentially lifted (SL) inequality and its strength

$$rx_j + \sum_{i \in [n] \setminus \{j\}} \alpha_i x_i \geq r(k+1) \tag{6}$$

Theorem (Wolsey, 1976)

The SL inequality (6) defines a facet of $\operatorname{conv}(\mathcal{X})$.

Complexity of computing the SL inequality

$$z_\ell = \max_{x \in \mathbb{Z}_+^{|\mathcal{S}|+2}} \quad \frac{r(k+1) - rx_j - \sum_{i \in \mathcal{S}} \alpha_i x_i}{x_\ell}$$
$$\text{s.t.} \quad a_j x_j + a_\ell x_\ell + \sum_{i \in \mathcal{S}} a_i x_i \geq b,$$
$$x_\ell \geq 1. \tag{7}$$

Theorem

The lifting problem (7) is NP-hard.

Solved by **dynamic programming** in $\mathcal{O}(nb)$.

- Fix $x_\ell \Rightarrow$ integer knapsack problem.
- Collect all the information calculated in the previous steps.

Bounds on the lifting coefficients

Theorem

Let $a_i = k_i a_j + r_i$ *for* $i \in [n] \backslash \{j\}$ *where* $k_i, r_i \in \mathbb{Z}_+$ *and* $1 \le r_i \le a_j$,

$$r x_j + \sum_{i \in [n] \backslash \{j\}} \alpha_i x_i \ge r(k+1) \tag{8}$$

be the SL inequality, and $\mathcal{T} = \{i \in [n] : r_i < r\}$. *Then*

(i) *if* $i \in \mathcal{T}$, $r k_i \le \alpha_i \le r(k_i + 1)$; *and*

(ii) *if* $i \in [n] \backslash (\mathcal{T} \cup \{j\})$, $\alpha_i = r(k_i + 1)$.

- Dimension reduction of the lifting problem.

$$r x_j + \sum_{i \in \mathcal{T}} \alpha_i x_i + \sum_{i \in [n] \backslash (\mathcal{T} \cup \{j\})} r(k_i + 1) x_i \ge r(k+1)$$

- For an LP relaxation solution x^*, no violated SL inequality (9) exists if

$$r x_j^* + \sum_{i \in \mathcal{T}} r k_i x_i^* + \sum_{i \in [n] \backslash (\mathcal{T} \cup \{j\})} r(k_i + 1) x_i^* \ge r(k+1).$$

Comparison with the mixed integer rounding (MIR) inequality

- **SL inequality**

$$rx_j + \sum_{i \in \mathcal{T}} \alpha_i x_i + \sum_{i \in [n] \backslash (\mathcal{T} \cup \{j\})} r(k_i + 1) x_i \geq r(k+1) \qquad (9)$$

  - If $i \in [n] \backslash (\mathcal{T} \cup \{j\})$, $\alpha_i = r(k_i + 1)$ where $\mathcal{T} = \{i \in [n] : r_i < r\}$.

- **MIR inequality** [Nemhauser-Wolsey, 1990], [Yaman, 2007]

$$rx_j + \sum_{i \in [n] \backslash \{j\}} (rk_i + \min\{r_i, r\}) x_i \geq r(k+1). \qquad (10)$$

  - Recognized as **the most effective cuts**.

- If $|\mathcal{T}| = 0$, the SL inequality is the same as the MIR inequality.

$\mathcal{T} = \{s\}$ is a singleton

(i) If $r = a_j$, then $\ell_s a_s \leq b r_s$ where ℓ_s denotes the least common multiple of a_j and r_s.

(ii) If $r < a_j$, then $\frac{r}{r_s} \in \mathbb{Z}$ and $r a_s \leq b r_s$.

Theorem

For $\mathcal{T} = \{s\}$, if conditions (i) and (ii) hold, the SL inequality is the same as the MIR inequality; otherwise, the SL inequality **strictly dominates** *the MIR inequality.*

Computational results: random instances

$$\min_{x \in \mathbb{Z}_+^n} \left\{ c^\top x \,:\, Ax \geq b \right\}.$$

- Sparsity of a constraint: $[0.05n, 0.15n]$
- $a_{ij} \in \{100, 101, \dots, 1000\}$
- $b = \lceil \frac{1}{2} A\boldsymbol{e} \rceil$
- $c = \boldsymbol{e}$

$n - m$	MIR				SL				MIR+SL			
	Gap	Solved	Time	Nodes	Gap	Solved	Time	Nodes	Gap	Solved	Time	Nodes
60 – 60	46.0	100	34.0	72874	49.9	100	16.2	37157	50.7	100	17.0	37624
70 – 70	33.5	75	1441.0	2569344	38.0	91	699.1	1590467	38.1	90	686.9	1551934

Gap $= 100 \cdot (z_{\text{ROOT}} - z_{\text{LP}})/(z_{\text{MIP}} - z_{\text{LP}})$, where

- z_{LP} is the objective value of the initial LP relaxation,
- z_{ROOT} is the objective value of the LP relaxation after adding cuts,
- z_{MIP} is the objective value of the optimal solution.

Computational results: real-world instances

- Staircase capacitated covering (SCC) problem, 20×10 instances, http://or.dei.unibo.it/library.
- Mixed pallet design (MPD) problem, 14×10 instances, http://www.bilkent.edu.tr/~alpersen/Mixed_Pallet.

Problems	DEFAULT			DEFAULT+SL		
	Solved	Time	Nodes	Solved	Time	Nodes
SCC	98	36.3	2128	100	28.3	1545
MPD	116	82.9	103751	126	30.3	30496

Extensions

- For 1065 instances in MIPLIB 2017, the SL cuts can be constructed for only 147 instances.
- The performance is neutral.
- More investigations on $\mathrm{conv}(\{x \in \mathbb{Z} : a^\top x \geq b,\ 0 \leq x \leq u\})$ and efficient aggregation procedures are needed.

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Thank you for your attention!