Games on Networks: Efficiency of Equilibria (A Very Biased View)

Marco Scarsini

LUISS

Games on networks

- Various classes of games are defined in terms of a network. For instance:
  - Network formation games.
  - Information transmission games.
  - Routing games.
  - Location games.
- For many of these games the efficiency of their equilibria has been studied.
- In particular the computer science literature has dealt with measures of inefficiency and their bounds.

Cost games

- A cost game $\Gamma = (\mathscr{N}, \mathscr{A}, \boldsymbol{c})$, where
  - $\mathscr{N} = \{1, \ldots, N\}$ is the set of players,
  - $\mathscr{A}_i$ is the set of actions of player i and $\mathscr{A} = \times_{i \in \mathscr{N}} \mathscr{A}_i$,
  - $c_i \colon \mathscr{A} \to \mathbb{R}$ is the cost function of player i and $\boldsymbol{c} = (c_1, \ldots, c_N)$.
- $\mathrm{SC}(\boldsymbol{a}) = \sum_{i \in \mathscr{N}} c_i(a_i)$ is the social cost of action profile $\boldsymbol{a}$.
- $\boldsymbol{a}^* \in \mathscr{A}$ is a Nash equilibrium if for all $i \in \mathscr{N}$ and all a_i

$$c_i(\boldsymbol{a}^*) \leq c_i(a_i, \boldsymbol{a}^*_{-i}).$$

- $\mathscr{E}(\Gamma)$ is the set of Nash equilibria of game Γ.
- $\mathrm{BEq}(\Gamma) = \min_{\boldsymbol{a} \in \mathscr{E}(\Gamma)} \mathrm{SC}(\boldsymbol{a})$ is the best equilibrium social cost.
- $\mathrm{WEq}(\Gamma) = \max_{\boldsymbol{a} \in \mathscr{E}(\Gamma)} \mathrm{SC}(\boldsymbol{a})$ is the worst equilibrium social cost.
- $\mathrm{Opt}(\Gamma) = \min_{\boldsymbol{a} \in \mathscr{A}} \mathrm{SC}(\boldsymbol{a})$ is the optimum social cost.

Measures of efficiency

- The price of anarchy (PoA) of game Γ is

$$\text{PoA}(\Gamma) = \frac{\text{WEq}(\Gamma)}{\text{Opt}(\Gamma)}.$$

- Koutsoupias and Papadimitriou (1999), Papadimitriou (2001).
- The price of stability (PoS) of game Γ is

$$\text{PoS}(\Gamma) = \frac{\text{BEq}(\Gamma)}{\text{Opt}(\Gamma)}.$$

- Schulz and Stier-Moses (2003), Anshelevich et al. (2008).

Congestion games

- A (finite) set $\mathscr{E}$ of resources, e.g., edges of a network.
- Achieving a certain goal requires the use of some subsets of these resources, e.g., edges that form a path from a source to a destination. Call $\mathscr{P}$ the set of feasible subsets.
- Agents are of different types. All the agents of the same type have the same weight and want to achieve the same goal, i.e., go from the same source to the same destination.
- A congestion game is a game where each agent can use a subset $p \in \mathscr{P}$ to achieve her goal and the cost of using a resource is a weakly increasing function on the number of agents who use it.
- Congestion games have been introduced by Rosenthal (1973).
- They have important uses in applications and interesting mathematical properties.

Symmetric congestion games

- A set $\mathscr{N}$ of players.
- A finite set $\mathscr{E}$ of resources.
- A set $\mathscr{P} \subset 2^{\mathscr{E}}$ of strategies.
- Call p_i the strategy of player i and $\boldsymbol{p} = (p_i)_{i \in \mathscr{N}}$ the strategy profile.
- For each $p \in \mathscr{P}$ a flow f_p that represents the number of players who choose strategy p:

$$f_p = \operatorname{card}\{i \in \mathscr{N} : p_i = p\}.$$

- For each $e \in \mathscr{E}$ a load

$$x_e = \sum_{\substack{p \ni e \\ p \in \mathscr{P}}} f_p$$

  that represents the number of agents who use resource e.
- For each $e \in \mathscr{E}$ a weakly increasing delay function $c_e(x_e)$.
- For each player i, a cost function $c^{(i)} : \mathscr{P}^N \to \mathbb{R}_+$

$$c^{(i)}(\boldsymbol{p}) = \sum_{e \in p_i} c_e(x_e).$$

Potential games

- Let $\mathscr{A} = \times_{i=1}^{N} \mathscr{A}_i$ and $\boldsymbol{c} = (c^{(1)}, \ldots, c^{(N)})$.
- A game $\langle \mathscr{N}, \mathscr{A}, \boldsymbol{c} \rangle$ is an exact potential game if there exists a function $\Psi : \mathscr{A} \to \mathbb{R}$ such that for all $a_{-i} \in \mathscr{A}_{-i}$, for all $a_i', a_i'' \in \mathscr{A}_i$

$$c^{(i)}(a_i', a_{-i}) - c^{(i)}(a_i'', a_{-i}) = \Psi(a_i', a_{-i}) - \Psi(a_i'', a_{-i}).$$

- Potential games admit pure Nash equilibria. They are the local minima (i.e., minima along each coordinate) of the potential function.
- Potential games have been studied by Monderer and Shapley (1996).

Congestion games and potential games

- Symmetric congestion games are potential games, therefore they admit pure Nash equilibria.

Theorem (Rosenthal (1973))

Let $\Gamma = (\mathscr{N}, \mathscr{P}, \boldsymbol{c})$ *be a symmetric congestion game. Then*

$$\Psi(\boldsymbol{p}) = \sum_{e \in \mathscr{E}} \sum_{i=1}^{x_e} c_e(i)$$

is a potential function for Γ.

Proof

Proof.

Given a strategy profile $\boldsymbol{p}$, if player i deviates from p_i to p_i', then

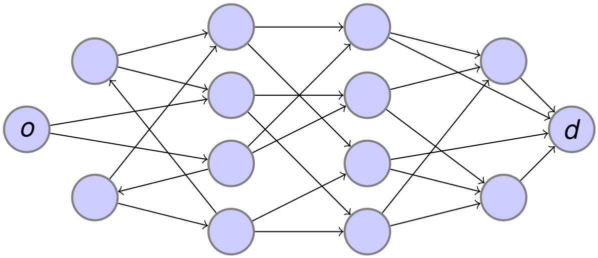
$$x_e' = \begin{cases} x_e + 1 & \text{if } e \in p_i' \setminus p_i, \\ x_e - 1 & \text{if } e \in p_i \setminus p_i', \\ x_e & \text{otherwise.} \end{cases}$$

$$\begin{aligned} \Psi(p_i', \boldsymbol{p}_{-i}) - \Psi(\boldsymbol{p}) &= \sum_{e \in p_i' \setminus p_i} c_e(x_e + 1) - \sum_{e \in p_i \setminus p_i'} c_e(x_e) \\ &= \sum_{e \in p_i'} c_e(x_e') - \sum_{e \in p_i} c_e(x_e) \\ &= c^{(i)}(p_i', \boldsymbol{p}_{-i}) - c^{(i)}(\boldsymbol{p}) \end{aligned}$$

□

Symmetric routing games

- A routing game is a special case of congestion game.
- A directed multigraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$.
- A single origin-destination (O/D) pair (o, d).
- Players choose paths from source to destination.
- In symmetric games all players have the same weight.



Nash equilibrium

- Let $\Gamma = (\mathscr{G}, \mathscr{P}, \boldsymbol{c})$ be a symmetric routing game.
- $\boldsymbol{p}^* \in \mathscr{A}$ is a Nash equilibrium of Γ if for all $i \in \mathscr{N}$ and all a_i

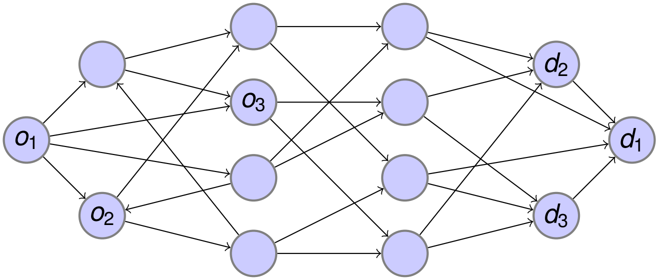
$$c_i(\boldsymbol{p}^*) \leq c_i(p_i, \boldsymbol{p}^*_{-i}),$$

- that is,

$$\sum_{e \in p_i^*} c_e(x_e) \leq \sum_{e \in p_i^* \cap p_i} c_e(x_e) + \sum_{e \in p_i \setminus p_i^*} c_e(x_e + 1).$$

- This last inequality is expressed just in term of loads.

Multiple O/D pairs



Smoothness

Definition

A game is (λ, μ)-smooth if for every $\boldsymbol{p}, \boldsymbol{p}' \in \mathscr{P}$

$$\sum_{i=1}^{N} c_i(p_i', \boldsymbol{p}_{-i}) \leq \lambda \, \mathrm{SC}(\boldsymbol{p}') + \mu \, \mathrm{SC}(\boldsymbol{p}).$$

Theorem (Roughgarden (2015))

Let $\lambda > 0$ and $\mu < 1$. If a game Γ is (λ, μ)-smooth, then

$$\mathrm{PoA}(\Gamma) \leq \frac{\lambda}{1 - \mu}.$$

Proof

Proof.

- Let $\boldsymbol{p}^*$ be a Nash equilibrium (NE) profile and $\tilde{\boldsymbol{p}}$ an optimum profile. Then

$$\begin{aligned} \mathrm{SC}(\boldsymbol{p}^*) = \sum_{i=1}^{N} c_i(\boldsymbol{p}^*) &\leq \sum_{i=1}^{N} c_i(\tilde{p}_i, \boldsymbol{p}^*_{-i}) \\ &\leq \lambda\, \mathrm{SC}(\tilde{\boldsymbol{p}}) + \mu\, \mathrm{SC}(\boldsymbol{p}^*) \end{aligned}$$

- First inequality by NE.
- Second inequality by (λ, μ)-smoothness.

□

Bound for the PoA

Theorem (Suri et al. (2007))

Let $\Gamma = (\mathscr{G}, \mathscr{P}, \boldsymbol{c})$ *be a symmetric routing game with affine costs. Then*

$$\mathrm{PoA}(\Gamma) \leq \frac{5}{2}.$$

The bound is tight.

Proof

Lemma (Christodoulou and Koutsoupias (2005))

For all $y, z \in \mathbb{N}$

$$y(z+1) \leq \frac{5}{3}y^2 + \frac{1}{3}z^2.$$

Proof, continued

Proof of Theorem.

For all $a, b \geq 0$ and $y, z \in \mathbb{N}$

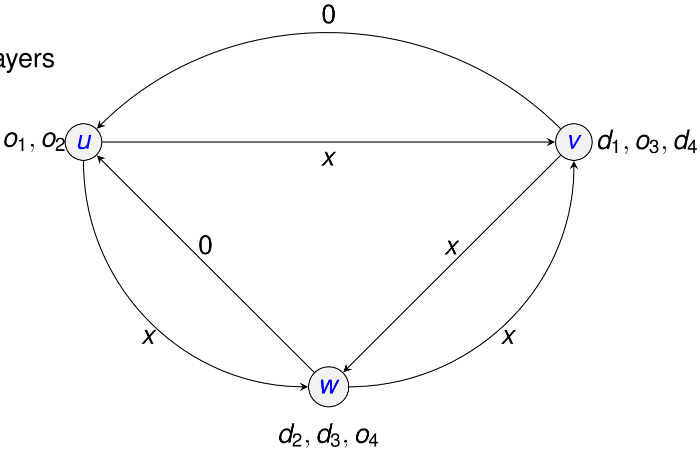
$$ay(z+1)+by \leq \frac{5}{3}(ay^2+by)+\frac{1}{3}(az^2+bz) = \frac{5}{3}(ay+b)y+\frac{1}{3}(az+b)z.$$

$$\begin{aligned}
\sum_{i=1}^{N} c_i(\tilde{p}_i, \boldsymbol{p}^*_{-i}) &\leq \sum_{e\in\mathscr{E}} (a_e(x_e^* + 1) + b_e)\tilde{x}_e \\
&\leq \sum_{e\in\mathscr{E}} \frac{5}{3}(a_e\tilde{x}_e + b_e)\tilde{x}_e + \sum_{e\in\mathscr{E}} \frac{1}{3}(a_e x_e^* + b_e)x_e^* \\
&= \frac{5}{3}\mathrm{SC}(\tilde{\boldsymbol{p}}) + \frac{1}{3}\mathrm{SC}(\boldsymbol{p}^*).
\end{aligned}$$

Hence the game is $\left(\frac{5}{3}, \frac{1}{3}\right)$-smooth. □

Proof: tightness

4 players



Tightness, continued

- Optimum. All players take a single-hop path.
- Best equilibrium. Same as optimum.
- Worst equilibrium. All players take a two-hop path.
- $\text{Opt} = \text{BEq} = 4$.
- $\text{WEq} = 10$.
- $\text{PoS} = 1, \quad \text{PoA} = 5/2$.

Symmetric games

Theorem (Correa et al. (2017))

The price of anarchy of a symmetric affine network routing games at most $5/2$.

Proof.

The result is obtained by considering a sequence of networks where both the number of paths and the number of players are increasing function of n and the letting $n \to \infty$. □

Large games

- When the number of players is large, although pure Nash equilibria are known to exist, computing them becomes daunting.
- It is customary to approximate a finite large network game with a nonatomic network game.
- The idea is to take the limit of a finite game as the number of players increases and their size gets smaller and smaller, in a way that the total mass of players is kept fixed.
- Haurie and Marcotte (1985), Milchtaich (2000), and Jaquot and Wan (2018) deal with similar problems.
- The limit game has a continuum of players, and the size of each of them is zero.
- Games with a continuum of players have been studied by Schmeidler (1973) and Mas-Colell (1984) among others. The required mathematical machinery is nontrivial.
- There is a simple way to treat nonatomic network routing games.

Nonatomic routing games

- A finite directed multi-graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with vertex set $\mathscr{V}$ and edge set $\mathscr{E}$.
- A finite set of O/D pairs (o^i, d^i), $i \in \mathscr{I}$.
- For $i \in \mathscr{I}$ a traffic demand $\mu^i \geq 0$.
- For $i \in \mathscr{I}$ a set $\mathscr{P}^i$ of (simple) paths joining o^i to d^i. The sets $\mathscr{P}^i$ are disjoint. $\mathscr{P} \equiv \bigcup_{i \in \mathscr{I}} \mathscr{P}^i$.

Nonatomic routing games, continued

- A set of feasible routing flows $\boldsymbol{f} = (f_p)_{p \in \mathscr{P}}$ in the network

$$\mathscr{F} = \left\{ \boldsymbol{f} \in \mathbb{R}_+^{\mathscr{P}} : \textstyle\sum_{p \in \mathscr{P}^i} f_p = \mu^i \text{ for all } i \in \mathscr{I} \right\}.$$

- A routing flow $\boldsymbol{f} \in \mathscr{F}$ induces a load on each edge $e \in \mathscr{E}$

$$x_e = \sum_{p \ni e} f_p.$$

- $\boldsymbol{x} = (x_e)_{e \in \mathscr{E}}$ is the load profile on the network.

Nonatomic routing games, continued

- A nondecreasing, continuous cost function $c_e \colon [0, \infty) \to (0, \infty)$ represents the latency experienced to traverse edge e.
- $c_e(x_e)$ is the delay on edge $e \in \mathscr{E}$ for a load profile $\boldsymbol{x} = (x_e)_{e \in \mathscr{E}}$ induced by a feasible routing flow $\boldsymbol{f} = (f_p)_{p \in \mathscr{P}}$.
- $$c_p(\boldsymbol{f}) \equiv \sum_{e \in p} c_e(x_e).$$
- $\Gamma = (\mathscr{G}, \mathscr{I}, \{\mu^i\}_{i \in \mathscr{I}}, \{\mathscr{P}^i\}_{i \in \mathscr{I}}, \{c_e\}_{e \in \mathscr{E}})$ is a (nonatomic) routing game.

Wardrop equilibrium

- A routing flow $\boldsymbol{f}^*$ is a Wardrop equilibrium (WE) of Γ if $c_p(\boldsymbol{f}^*) \leq c_{p'}(\boldsymbol{f}^*)$ for all $p, p' \in \mathscr{P}^i$ such that $\boldsymbol{f}_p^* > 0$, for all $i \in \mathscr{I}$.

Characterization of WE

- WE are solutions of the (convex) minimization problem:

$$\begin{aligned} \text{minimize} \quad & \sum_{e \in \mathscr{E}} C_e(x_e), \\ \text{subject to} \quad & x_e = \sum_{p \ni e} f_p, \;\; \boldsymbol{f} \in \mathscr{F}, \end{aligned} \tag{WE}$$

  where $C_e(x_e) = \int_0^{x_e} c_e(w)\, \mathrm{d}w$ denotes the primitive of c_e.
- WE satisfy

$$\sum_{e \in \mathscr{E}} c_e(x_e^*)(x_e - x_e^*) \geq 0 \quad \text{for all } \boldsymbol{f} \in \mathscr{F}.$$

- All WE have the same social cost.

Social optimum

- A socially optimum (SO) flow $\tilde{\boldsymbol{f}}$ is a solution to the total cost minimization problem:

$$\begin{aligned} \text{minimize} \quad & \mathrm{SC}(\boldsymbol{f}) = \sum_{p \in \mathscr{P}} f_p c_p(\boldsymbol{f}), \\ \text{subject to} \quad & \boldsymbol{f} \in \mathscr{F}. \end{aligned} \tag{SO}$$

- Equivalently

$$\begin{aligned} \text{minimize}_{\boldsymbol{x} \in \mathscr{X}} \quad & \mathrm{SC}(\boldsymbol{x}) = \sum_{e \in \mathscr{E}} x_e c_e(x_e), \\ \text{where} \quad & \mathscr{X} = \{\boldsymbol{x} : x_e = \sum_{p \ni e} f_p, \boldsymbol{f} \in \mathscr{F}\}. \end{aligned} \tag{SO}$$

Characterization of SO

- Define $\tilde{c}_e(x) = c_e(x) + x c_e'(x)$.
- Let for every $e \in \mathscr{E}$, the function $x \mapsto x c_e(x)$ be convex. Then $\tilde{\boldsymbol{f}}$ is an equilibrium of the game $\tilde{\Gamma} = (\mathscr{G}, \mathscr{I}, \{\mu^i\}_{i \in \mathscr{I}}, \{\mathscr{P}^i\}_{i \in \mathscr{I}}, \{\tilde{c}_e\}_{e \in \mathscr{E}})$.

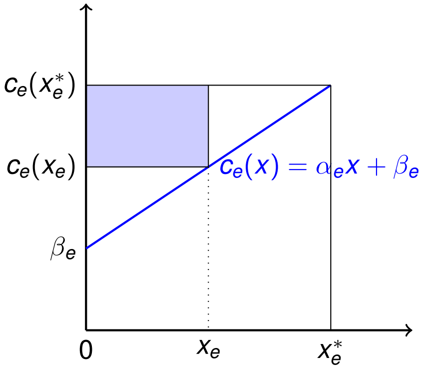
Roughgarden and Tardos

- Roughgarden and Tardos (2002) showed that the PoA in (nonatomic) routing games with affine costs never exceeds $4/3$, irrespective of the network's topology.
- If the cost functions are polynomials of degree d or less, the worst-case value of the PoA grows as $\Theta(d/\log d)$, (Roughgarden (2003)).
- Hence, the selfish routing can be arbitrarily bad in networks with polynomial costs.
- Given the typically nonlinear relation between traffic loads and travel times, the intervention of a central planner seems necessary in order to regain some efficiency.

A geometric look at PoA

Theorem (Roughgarden and Tardos (2002))

The PoA in games with affine costs is at most $4/3$.



Proof (Correa et al. (2008))

Proof.

$$
\begin{aligned}
\mathrm{SC}(\boldsymbol{x}^*) &= \sum_{e \in \mathscr{E}} c_e(x_e^*) x_e^* \\
&\leq \sum_{e \in \mathscr{E}} c_e(x_e^*) x_e \\
&= \sum_{e \in \mathscr{E}} c_e(x_e) x_e + \sum_{e \in \mathscr{E}} \left(c_e(x_e^*) - c_e(x_e) \right) x_e \\
&\leq \mathrm{SC}(\boldsymbol{x}) + \frac{1}{4} \sum_{e \in \mathscr{E}} c_e(x_e^*) x_e^* \\
&= \mathrm{SC}(\boldsymbol{x}) + \frac{1}{4} \mathrm{SC}(\boldsymbol{x}^*)
\end{aligned}
$$

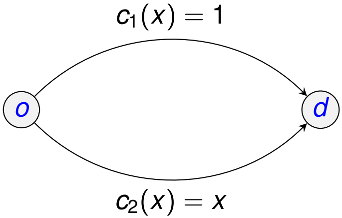
□

Similar proof

- A similar technique can be used to prove bounds for the PoA when costs are polynomial.

Pigou's model (1920)

Pigou was probably the first to point out the inefficiencies of selfish routing.



- If the total demand is $\mu = 1$, in equilibrium all the flow goes to the bottom edge, everybody experiences a delay of 1, and the total cost is 1.
- A central planner could do better by splitting the travelers equally among the two roads, achieving an optimal cost of $3/4$.
- Hence PoA$= 4/3$.

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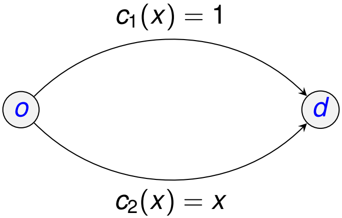

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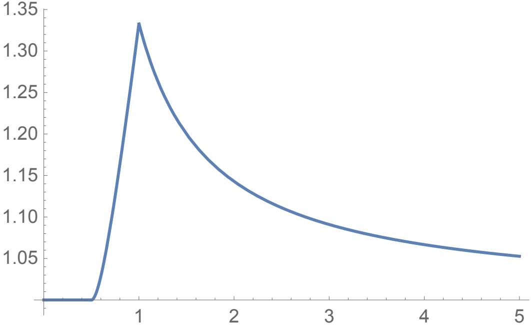
What if the total demand is not 1?

Pigou, continued

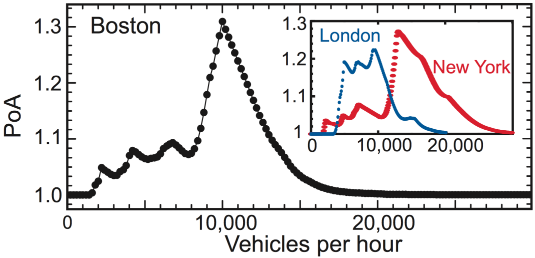
Price of Anarchy for the Pigou model, as a function of the demand



- Is this behavior of PoA general, or is it specific to the Pigou model?
- Is the issue relevant?

High congestion

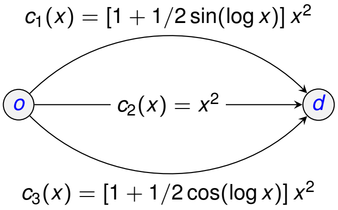
- Youn et al. (2008), O'Hare et al. (2016) show that the PoA is usually close to 1 for very high and very low traffic, and it fluctuates in the intermediate regime.



Is it always true?

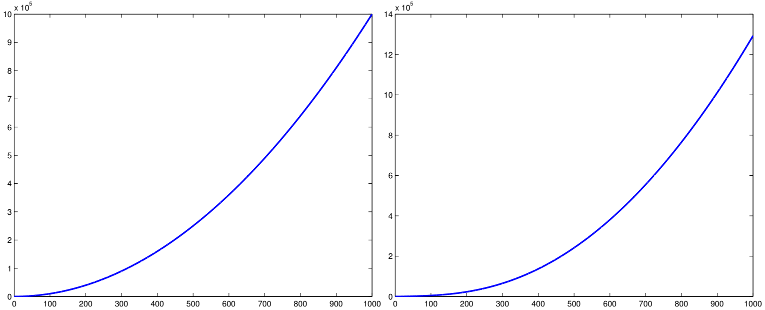
- Is it always the case that the PoA goes to one, as the demand increases?
- Is it at least true for single O/D networks?
- Is it at least true for parallel networks?
- Is it true for well-behaved, e.g., convex, cost functions?

Counterexample (Colini-Baldeschi et al. (2017))



- Single O/D.
- Parallel network.
- Convex cost functions.

Cost functions



Periodic behavior

Figure: The PoA is bounded away from 1 and periodic on a logarithmic scale.

Single O/D with polynomial costs

Theorem

If the network has a single O/D and the costs are polynomial, then

$$\lim_{\mu \to 0} \mathrm{PoA}(\Gamma_\mu) = 1,$$

$$\lim_{\mu \to \infty} \mathrm{PoA}(\Gamma_\mu) = 1.$$

Regularly varying functions

Definition

A function $g\colon [0,\infty) \to (0,\infty)$ is regularly varying at ω ($\omega = 0$ or $\omega = \infty$) if

$$h(x) = \lim_{t \to \omega} \frac{g(tx)}{g(t)} \quad \text{is finite and nonzero for all } x \geq 0.$$

- Standard examples of regularly varying functions include all affine, polynomial and logarithmic/polylogarithmic functions.
- The notion itself dates back to the work of Karamata (1930, 1933) and has been used extensively in probability and large deviations theory.

Benchmark functions & Tightness

- A regularly varying $c\colon (0,\infty) \to (0,\infty)$ is called a benchmark for Γ_μ at ω if the following (possibly infinite) limits exist for all edges $e \in \mathscr{E}$

$$\alpha_e = \lim_{x \to \omega} \frac{c_e(x)}{c(x)}.$$

An edge e is fast, slow, or tight relative to c if α_e is 0, ∞, or in-between.

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- The c-index of a path $p \in \mathscr{P}$ is determined by its slowest edge

$$\alpha_p = \max_{e\in p} \alpha_e.$$

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- The c-index of a path $p \in \mathscr{P}$ is determined by its slowest edge

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- The fastest paths are those with smallest c-index

$$\alpha = \min_{p \in \mathscr{P}} \alpha_p,$$

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- The fastest paths are those with smallest c-index

$$\alpha = \min_{p\in\mathscr{P}} \alpha_p,$$

- When $0 < \alpha < \infty$ we say that the benchmark is tight, and then we say that a network is tight if it admits a tight benchmark.

Single O/D: main results

Theorem

Let Γ_μ be a nonatomic routing game with a single O/D pair.

- *If the network is tight under light traffic ($\omega = 0$), then*

$$\lim_{\mu \to 0} \text{PoA}(\Gamma_\mu) = 1.$$

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- *If the network is tight under light traffic ($\omega = 0$), then*

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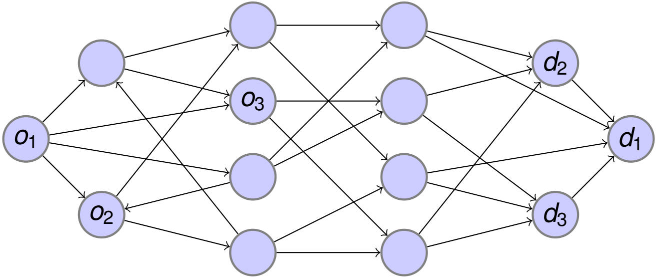
- *If the network is tight under heavy traffic ($\omega = \infty$), then*

$$\lim_{\mu \to \infty} \text{PoA}(\Gamma_\mu) = 1.$$

Multiple O/D pairs

A family of O/D pairs $i \in \mathscr{I}$ with demands μ^i to be routed from origin o_i to destination d_i using the paths $p \in \mathscr{P}^i$.



Total demand $\mu = \sum_{i \in \mathscr{I}} \mu^i$

Multiple O/D: high congestion

- If there is a single O/D pair, the network becomes highly congested when the inflow grows to infinity.
- If there are several O/D pairs, the traffic inflow of each pair could be growing at very different rates.
- In particular, the inflow of some O/D pairs could remain finite (or even vanish), but the network may still become heavily congested if the aggregate demand grows large.

Multiple O/D: high congestion

- The total traffic demand in the network is

$$\mu = \sum_{i \in \mathscr{I}} \mu^i,$$

- The relative inflow of the i-th O/D pair is

$$\lambda^i = \mu^i / \mu.$$

- High congestion refers to the limit $\mu \to \infty$, with no assumptions on the behavior of the relative inflow vector $\lambda = (\lambda^i)_{i \in \mathscr{I}}$ in this limit.

Tightness – Multiple O/D

- Define

$$\alpha^i = \min_{p \in \mathscr{P}^i} \alpha_p,$$

$$\alpha = \max_{i \in \mathscr{I}} \alpha^i,$$

- The network is tight if $0 < \alpha < \infty$.

Multiple O/D: main results

Theorem

Consider a multiple O/D network with fixed relative inflow rates λ^i *and a variable total demand* μ*. If the network is tight at* ω*, then*

$$\lim_{\mu \to \omega} \mathrm{PoA}(\Gamma_\mu) = 1.$$

Multiple O/D: main results

Theorem

Consider a multiple O/D network with fixed relative inflow rates λ^i and a variable total demand μ. If the network is tight at ω, then

$$\lim_{\mu \to \omega} \mathrm{PoA}(\Gamma_\mu) = 1.$$

- Tightness requires that every O/D pair has a path which is not slow, and that at least one O/D pair is tight.
- This is considerably weaker than asking every O/D pair to be tight, so the conditions under which the price of anarchy converges to 1 are very lax.

Variable inflow rates

- Let Γ_n be a sequence of nonatomic routing games with total demand $\mu_n = \sum_{i \in \mathscr{I}} \mu_n^i$ induced by a sequence of inflow rates μ_n^i for each $i \in \mathscr{I}$.
- The relative inflow rates $\lambda_n^i = \mu_n^i / \mu_n$ could now exhibit very different behaviors as $n \to \infty$.

Variable inflow rates

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Definition

A subset $\mathscr{I}' \subseteq \mathscr{I}$ of O/D pairs is called salient if the aggregate fraction of the traffic generated by the O/D pairs in $\mathscr{I}'$ is non-negligible in the limit, namely

$$\liminf_{n \to \infty} \sum_{i \in \mathscr{I}'} \lambda_n^i > 0. \tag{1}$$

Variable inflow rates, continued

Theorem

Let Γ_n *be a sequence of nonatomic routing games with inflow rates* μ_n^i, *and total inflow* $\mu_n = \sum_{i \in \mathscr{I}} \mu_n^i$. *Suppose that:*

- *Traffic is either light or heavy in the limit, i.e.,* $\lim_{n \to \infty} \mu_n = \omega \in \{0, \infty\}$.
- *Every O/D pair has a path which is not slow, i.e.,* $\alpha^i < \infty$ *for all* $i \in \mathscr{I}$.
- *The set of tight O/D pairs is salient, i.e.,* $\liminf_{n \to \infty} \sum_{i : \alpha^i > 0} \lambda_n^i > 0$.

Then $\mathrm{PoA}(\Gamma_n) \to 1$ *as* $n \to \infty$.

Takeaways

- Inefficiency of equilibria can be measured with the PoA.
- Bounds for the PoA can be obtained for routing games.
- These bounds are different for atomic and nonatomic games.
- They do not depend on the topology of the network.
- They depend on the cost functions and on the demand.
- In nonatomic routing games for a large class of cost functions the PoA approaches 1 both in light and heavy traffic.

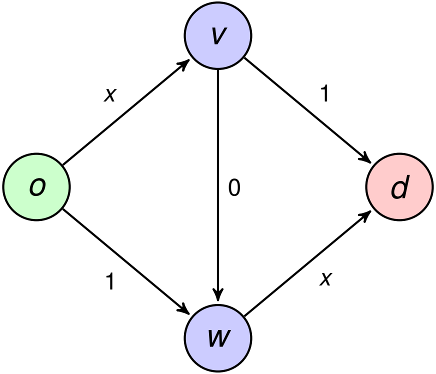
Open problems

- Can something be said for more general classes of cost functions, e.g., exponential?
- What is the behavior of the PoA in the region of medium traffic?
- What happens in atomic games?
- Can the framework be generalized to other classes of games?

How to restore efficiency?

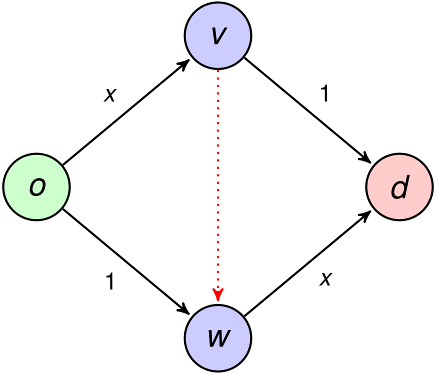
- How could a planner devise mechanisms that restore efficiency to the system?
- The idea is to modify the game in such a way that the equilibrium of the new game coincides with the optimum of the original game.
- This can (sometimes) be achieved in different ways.
- Some of them counterintuitive.
- For instance destroying some edge in the network.

Braess's paradox



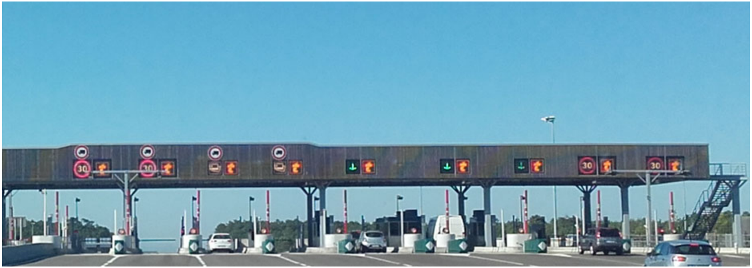
$$\mu = 1, \quad \text{Opt} = 3/2, \quad \text{Eq} = 2$$

Braess's paradox, continued



$$\mu = 1, \quad \text{Opt} = \text{Eq} = 3/2$$

Implementing Braess's paradox



The elimination of the $v \to w$ edge could be achieved by imposing a very high toll on that edge.

Tolls

- $\boldsymbol{\tau} = (\tau_e)_{e \in \mathscr{E}} \in \mathbb{R}^{\mathscr{E}}$ a toll vector.
- We allow for both positive and negative tolls.
- $c_e^{\boldsymbol{\tau}}$ the cost of edge e under the toll $\boldsymbol{\tau}$, $c_e^{\boldsymbol{\tau}}(x_e) := c_e(x_e) + \tau_e$.
- $c_p^{\boldsymbol{\tau}}(\boldsymbol{f}) := \sum_{e \in p} c_e^{\boldsymbol{\tau}}(x_e)$.
- $\Gamma^{\boldsymbol{\tau}} := (\mathscr{G}, \mathscr{I}, \boldsymbol{c}^{\boldsymbol{\tau}})$.
- $\tilde{\mathscr{P}}^i = \{p \in \mathscr{P}^i : \tilde{\boldsymbol{f}}_p(\boldsymbol{\mu}) > 0$ for some $\boldsymbol{\mu}$ and corresponding SO $\tilde{\boldsymbol{f}}(\boldsymbol{\mu})\}$.

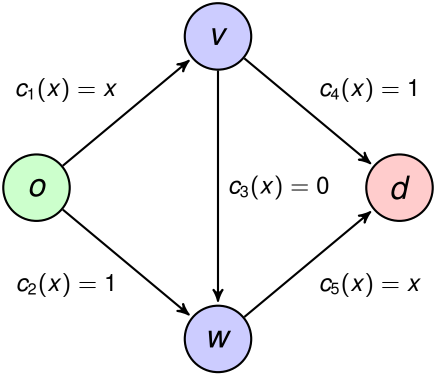
Optimal tolls

- Can we impose tolls on the edges of the network in such a way that the equilibrium flow of the game with tolls is socially optimal for the original game?
- The problem and its solution go back to by Pigou (1920) and Knight (1924).
- If the demand vector μ is known, tolls can be chosen as

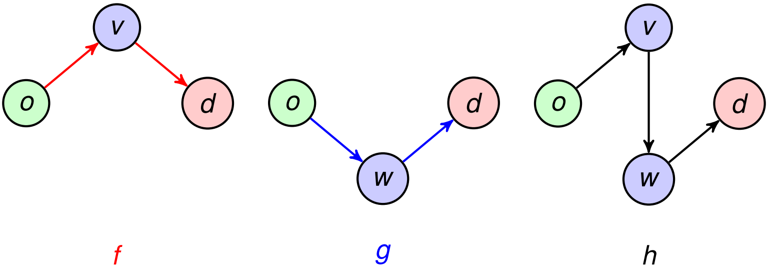
$$\tau_e = \tilde{x}_e c'_e(\tilde{x}_e).$$

- This way the optimum of the game $\Gamma = (\mathscr{G}, \mathscr{I}, \boldsymbol{c})$ will be obtained as an equilibrium of the game $\Gamma^{\tau} := (\mathscr{G}, \mathscr{I}, \boldsymbol{c}^{\tau})$.
- What if the demand μ is not known and is not correctly estimated?

Robustness



Flows



$$f + g + h = \mu$$

Optima and equilibria

μ	$\tilde{f}$	$\tilde{g}$	$\tilde{h}$	f^*	g^*	h^*	PoA
$[0, \frac{1}{2}]$	0	0	μ	0	0	μ	1
$[\frac{1}{2}, 1]$	$\mu - \frac{1}{2}$	$\mu - \frac{1}{2}$	$1 - \mu$	0	0	μ	$2\mu^2/(2\mu - \frac{1}{2})$
$[1, 2]$	$\frac{\mu}{2}$	$\frac{\mu}{2}$	0	$\mu - 1$	$\mu - 1$	$2 - \mu$	$4/(\mu + 2)$
$[2, \infty)$	$\frac{\mu}{2}$	$\frac{\mu}{2}$	0	$\frac{\mu}{2}$	$\frac{\mu}{2}$	0	1

Misestimation

- If $\mu = 1$, then the optimal toll vector is

$$\tau_1 = \tau_5 = 1/2, \quad \tau_2 = \tau_3 = \tau_4 = 0.$$

- Suppose that the planner estimates μ to be 1, so she uses the above tolls, but that actually $\mu = 1/2$.
- The equilibrium flow for the game with tolls is $f = g = 1/4$, $h = 0$.
- Then the social cost in equilibrium for the game with tolls is

$$\frac{1}{4}\left(1 + \frac{1}{4} + \frac{1}{2}\right)2 = \frac{7}{8} = \frac{5}{8} + \frac{1}{4},$$

  where $\frac{1}{4}$ is the toll.
- Without the toll the equilibrium social cost would have been $\frac{1}{2} < \frac{5}{8}$.

Demand-independent optimal tolls

Definition

$\boldsymbol{\tau} \in \mathbb{R}^{\mathscr{E}}$ is called *demand-independent optimal toll* (DIOT) for Γ if for every demand vector $\boldsymbol{\mu} \in \mathbb{R}_{+}^{\mathscr{I}}$ every corresponding equilibrium with tolls $\boldsymbol{f}^{\boldsymbol{\tau}}(\boldsymbol{\mu}) \in \mathrm{Eq}(\Gamma^{\boldsymbol{\tau}})$ is optimal for Γ.

Definition

We call $\mathscr{C}_{\mathrm{BPR}}(\beta)$ the class of cost functions of the form

$$c_e(x) = t_e + a_e x_e^{\beta} \quad \text{for all } e \in \mathscr{E}.$$

DIOTs exist for BPR

Theorem (Colini-Baldeschi et al. (2018))

Consider a game $\Gamma = (\mathscr{G}, \mathscr{I}, \boldsymbol{c})$ with $c_e \in \mathscr{C}_{\mathrm{BPR}}(\beta)$ for all $e \in \mathscr{E}$. Let $\boldsymbol{\tau}$ be a toll vector such that

$$\sum_{e \in p} \left(\tau_e + \frac{\beta}{\beta+1} t_e \right) \leq \sum_{e \in p'} \left(\tau_e + \frac{\beta}{\beta+1} t_e \right).$$

for all $i \in \mathscr{I}$ and all $p \in \tilde{\mathscr{P}}^i$ and all $p' \in \mathscr{P}^i$.
Then, $\boldsymbol{\tau}$ is a DIOT.

DIOTs exist for BPR

Theorem (Colini-Baldeschi et al. (2018))

Consider a game $\Gamma = (\mathscr{G}, \mathscr{I}, \boldsymbol{c})$ *with* $c_e \in \mathscr{C}_{\mathrm{BPR}}(\beta)$ *for all* $e \in \mathscr{E}$. *Let* $\boldsymbol{\tau}$ *be a toll vector such that*

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for all $i \in \mathscr{I}$ *and all* $p \in \tilde{\mathscr{P}}^i$ *and all* $p' \in \mathscr{P}^i$.
Then, $\boldsymbol{\tau}$ *is a DIOT.*

Corollary

Consider a game $\Gamma = (\mathscr{G}, \mathscr{I}, \boldsymbol{c})$ *with BPR-type cost functions. Then there exists a DIOT for* Γ.

Necessary conditions

Theorem

Consider a game $\Gamma = (\mathscr{G}, \mathscr{I}, \boldsymbol{c})$ *with BPR-type cost functions. If* $\boldsymbol{\tau}$ *is a DIOT for* Γ*, then*

$$\sum_{e \in p} \left(\tau_e + \tfrac{\beta}{\beta+1} t_e \right) = \sum_{e \in p'} \left(\tau_e + \tfrac{\beta}{\beta+1} t_e \right).$$

for all $i \in \mathscr{I}$ *and all* $p, p' \in \tilde{\mathscr{P}}^i$*.*

Only BPR

Theorem

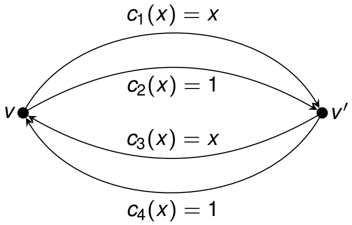
Let c be twice continuously differentiable, strictly semi-convex and strictly increasing, but not of BPR-type. Then there is a congestion game $\Gamma = (\mathscr{G}, \mathscr{I}, \boldsymbol{c})$ *with two parallel edges and cost functions* $c(x)$ *and* $c(x) + t$ *for some* $t \in \mathbb{R}_+$ *that does not have a DIOT.*

Non-negative tolls

Theorem

Consider a game $\Gamma = (\mathscr{G}, \mathscr{I}, \boldsymbol{c})$ with BPR-type cost functions where $\mathscr{G}$ is a directed acyclic multi-graph (DAMG). Then there exists a non-negative DIOT for Γ.

DAMG is not necessary



This graph is not a DAMG, but the following non-negative toll is a DIOT:

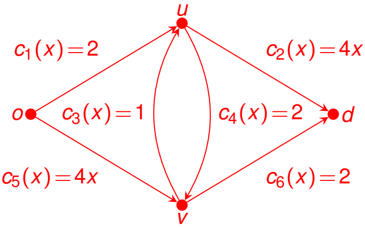
$$\tau_1 = \frac{1}{2} \quad \tau_2 = 0, \quad \tau_3 = \frac{1}{2}, \quad \tau_4 = 0.$$

Counterexample

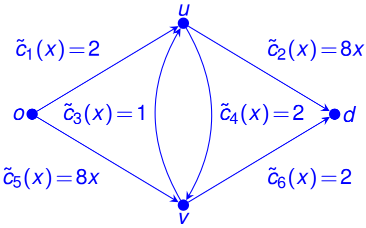
Proposition

There are networks with affine costs that do not admit a non-negative DIOT.

Counterexample, continued



(a) Cost functions.



(b) Marginal cost functions.

Figure: Cost functions c and marginal cost functions $\tilde{c}$ for a cyclic network.

Counterexample, continued

- For each path $p \in \mathscr{P}$, there exists a value μ of the demand such that p is used in the system optimum, hence $\tilde{\mathscr{P}} = \mathscr{P}$.
- For any DIOT $\boldsymbol{\tau}$,

$$\sum_{e \in p} t_e/2 + \tau_e = \sum_{e \in p'} t_e/2 + \tau_e \text{ for all all } p, p' \in \tilde{\mathscr{P}}^i.$$

-

$$\begin{aligned} T &= \frac{1}{2} + \tau_5 + \tau_3 + \tau_2, \\ T &= 3 + \tau_1 + \tau_4 + \tau_6, \\ T &= 1 + \tau_1 + \tau_2, \\ T &= 1 + \tau_5 + \tau_6. \end{aligned}$$

- $0 = \frac{3}{2} + \tau_3 + \tau_4$.
- Either τ_3 or τ_4 must be negative.

Positive revenue

Theorem

Consider a game $\Gamma = (\mathscr{G}, \mathscr{I}, \boldsymbol{c})$ *with BPR-type cost functions. If there exists an order* $\prec$ *on* $\mathscr{V}$ *such that for all* $i \in \mathscr{I}$ *we have* $o^i \prec d^i$*, then there exists a DIOT* $\boldsymbol{\tau}$ *that satisfies*

$$\sum_{e \in \mathscr{E}} \tau_e x_e \geq 0, \quad \text{for any feasible flow } \boldsymbol{f}. \qquad (*)$$

Corollary

Consider a game $\Gamma = (\mathscr{G}, \mathscr{I}, \boldsymbol{c})$ *with BPR-type cost and a single O/D pair. Then, there exists a DIOT* $\boldsymbol{\tau}$ *that satisfies (*).*

Takeways

- Tolls can be used to achieve efficiency.
- Optimal tolls in general depend on the demand.
- DIOTs exist for nonatomic routing games with bureau of public roads (BPR) cost functions.
- If the network is a DAMG, then the DIOT can be chosen to be nonnegative.
- Under the weaker condition that there exists an order of the vertices such that each origin precedes the corresponding destination, there exist a DIOT whose total revenue is nonnegative.

Open problems

- When costs are not BPR can approximately optimal tolls be computed?
- How good (or bad) would they be?
- What if the game is atomic?
- What if tolls can be exacted only on a subset of edges?
- What if tolls are exacted on vertices?

General social costs

- In several situations it is natural to consider social costs that are not the sum of the individual costs of the players.
- For instance Vetta (2002) introduces a class of games called utility games such that:
  - Each player i has a set of strategies A_i.
  - Call $\alpha_i(s)$ the welfare of player i.
  - If $A = \cup_i A_i$, a social welfare function $V(S)$ is defined for each $S \subset A$.
  - V is submodular.
  - $\sum_i \alpha_i(s) \leq V(s)$.
  - $\alpha_i(s) \leq V(s) - V(s_{-i})$.

Bound on PoA

Theorem (Vetta (2002))

If V is monotone, then the PoA of the utility game is at most 2.

- A generalization of the classical location game à la Hotelling is similar, but does not fall under the same umbrella.

Hotelling beach. Two sellers

- Two ice-cream sellers on a beach.
- Consumers uniformly distributed on the beach.
- They buy from the closer seller.
- Where should the two sellers stand?

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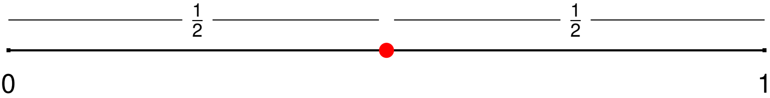

- In equilibrium they both stand in the middle.

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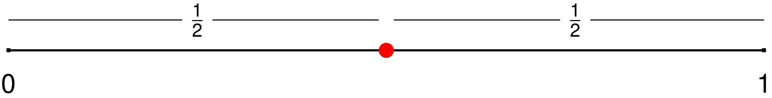

- In equilibrium they both stand in the middle.
- What if instead of living on a beach, consumers and sellers live on a network?

Buyers and sellers

Hypotheses on buyers

- Continuum of buyers, distributed on a network.
- Each one wants to buy one unit of a particular good whose price is fixed: they shop to the closest location.

Buyers and sellers

Hypotheses on buyers

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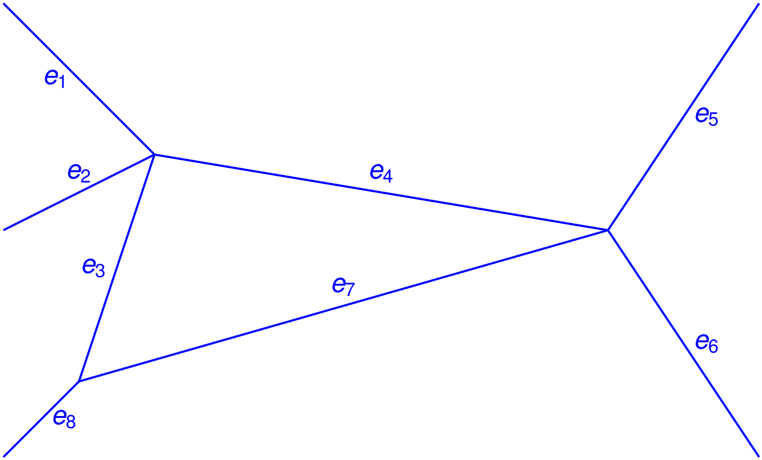
Hypotheses on sellers

- A finite number of sellers sell the same good at the same price.
- They simultaneously choose their locations.
- They want to sell as much as possible.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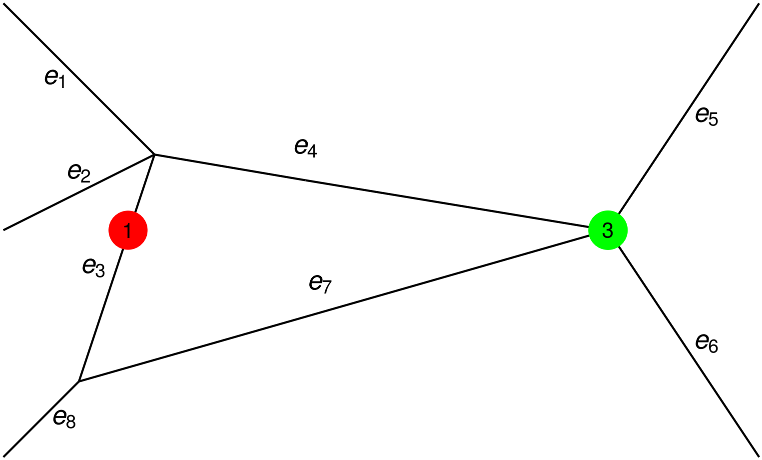
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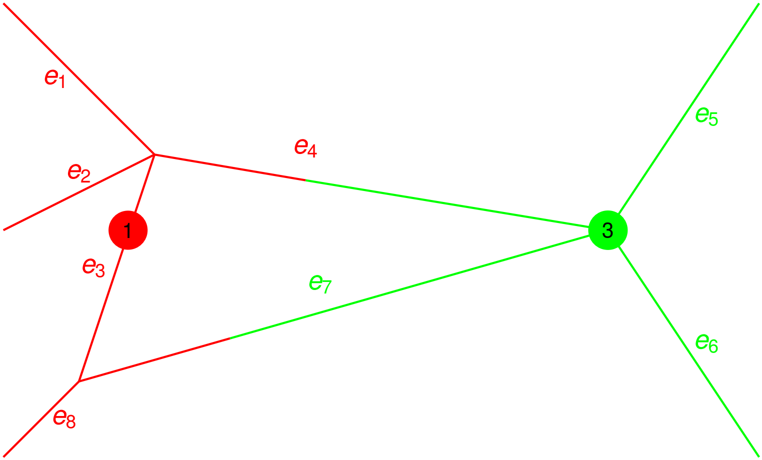
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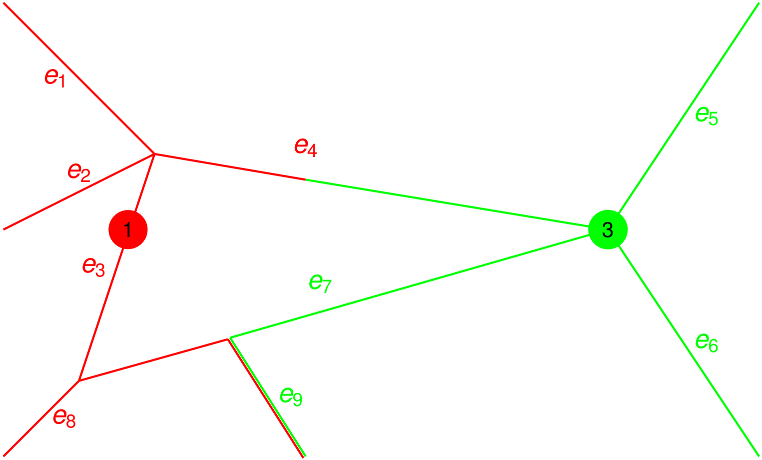
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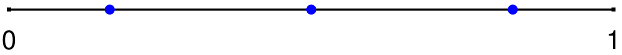
Generalized Hotelling game

- A normal form game $\mathscr{H}(n, S, \boldsymbol{\rho})$.
- A finite set $N = \{1, \ldots, n\}$ of players (sellers).
- The same action set S (the network) for each seller.
- The payoff ρ_i of seller i is the amount of consumers who shop at her store.

What kind of game?

- Is this a nice game?
- It is not a potential game.
- There is no pure equilibrium for 3 players in the unit interval.



- It is not possible to use general results to find (pure) equilibria of this game.

Existence of equilibria

- Pure equilibria of the generalized Hotelling game exist when the number of sellers is large enough.

Theorem (Fournier and S (2018))

For an arbitrary S, there exists $\bar{n} \in \mathbb{N}$ such that for every $n \geq \bar{n}$, the game $\mathscr{H}(n, S, \rho)$ admits a pure Nash equilibrium. In particular

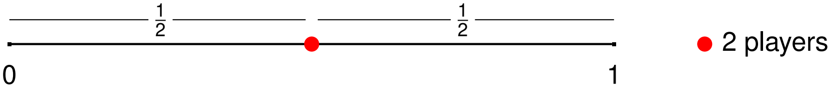
$$\bar{n} = 3\,\mathrm{card}(E) + \sum_{e \in E} \left\lceil \frac{5\lambda(e)}{\lambda^\star} \right\rceil,$$

where $\lambda^\star = \min_{e \in E} \lambda(e)$ is the length of the shortest edge.

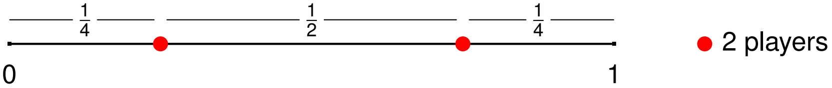
Back to the beach

The unit interval (Hotelling's beach)

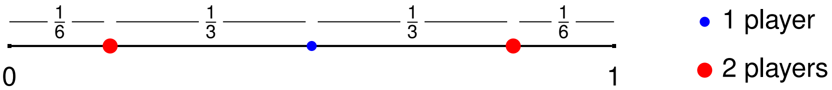
- For $n = 2$ there exists a unique pure Nash equilibrium.
- For $n = 3$ there is no pure Nash equilibrium.
- For $n = 4, 5$ there exists a unique pure Nash equilibrium.
- For $n \geq 6$ there exist infinitely many pure Nash equilbria.



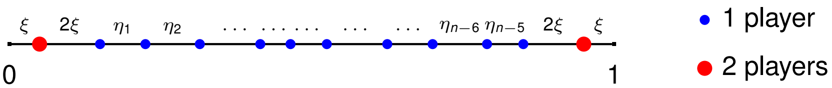
Unique equilibrium with 2 players.



Unique equilibrium with 4 players.

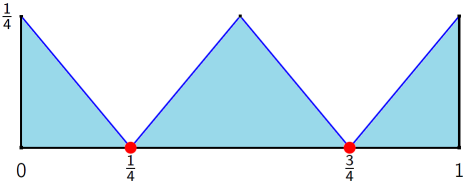


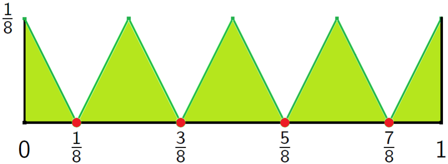
Unique equilibrium with 5 players.



Example of equilibrium with n players.

Efficiency of equilibria





Equilibrium social cost: $\frac{1}{8}$ Optimum social cost: $\frac{1}{16}$

- Call $\mathscr{E}_n$ the set of equilibria of $\mathscr{H}(n, S, \boldsymbol{\rho})$.
- For a strategy profile $\boldsymbol{x} \in S^n$, the social cost is

$$\sigma(\boldsymbol{x}) := \int_S \min_{i \in \{1,\dots,n\}} d(x_i, y)\, \mathrm{d}y.$$

- The price of anarchy is

$$\mathrm{PoA}(n) := \frac{\max_{\boldsymbol{x} \in \mathscr{E}_n} \sigma(\boldsymbol{x})}{\min_{\boldsymbol{x} \in S^n} \sigma(\boldsymbol{x})}.$$

- The price of stability is

$$\mathrm{PoS}(n) := \frac{\min_{\boldsymbol{x} \in \mathscr{E}_n} \sigma(\boldsymbol{x})}{\min_{\boldsymbol{x} \in S^n} \sigma(\boldsymbol{x})}.$$

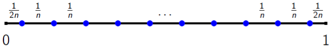


Figure: Social optimum $\bar{x}$ with n players.

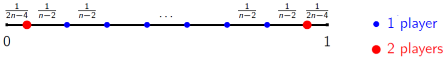


Figure: Best equilibrium $\check{x}$ with n players.

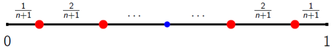


Figure: Worst equilibrium $\hat{x}$ with n players (n odd)

Figure: Worst equilibrium $\hat{x}$ with n players (n even).

On the unit interval

$$\mathrm{PoA}(n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 2\left(\dfrac{n}{n+1}\right) & \text{if } n > 3 \text{ is odd.} \end{cases}$$

For $n \geq 4$

$$\mathrm{PoS}(n) = \frac{n}{n-2}$$

Theorem

For any possible network

- $\text{PoA}(n) \to 2$ *as* $n \to \infty$,
- $\text{PoS}(n) \to 1$ *as* $n \to \infty$.

- For finite n we can have $\text{PoA}(n) \geq 2$.
- Consider a Hotelling game $\mathscr{H}(9, S_3, \boldsymbol{\rho})$.
- There is a unique equilibrium $\boldsymbol{x}^*$ such that

$$\sigma(\boldsymbol{x}^*) = 9\left(\frac{1}{3}\right)^2 \frac{1}{2} = \frac{1}{2}.$$

- There exists a strategy profile $\boldsymbol{x}^\sharp$ such that

$$\sigma(\boldsymbol{x}^\sharp) = 2 \cdot 7\left(\frac{1}{7}\right)^2 \frac{1}{2} + 5\left(\frac{1}{5}\right)^2 \frac{1}{2} = \frac{17}{70}.$$

- Therefore

$$\text{PoA}(9) \geq \frac{1}{2} \cdot \frac{70}{17} = \frac{35}{17} > 2.$$

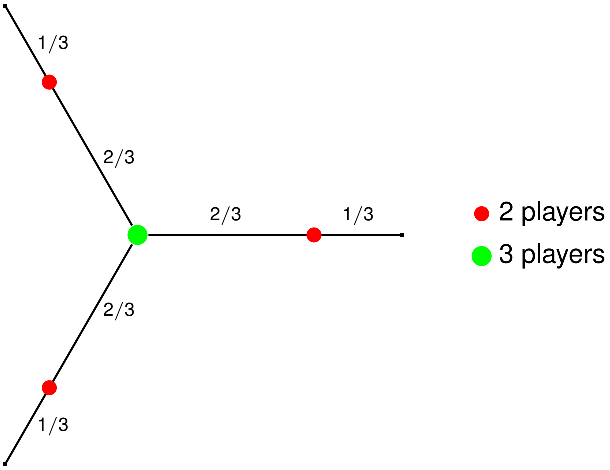


Figure: Equilibrium on S_3 with 9 players.

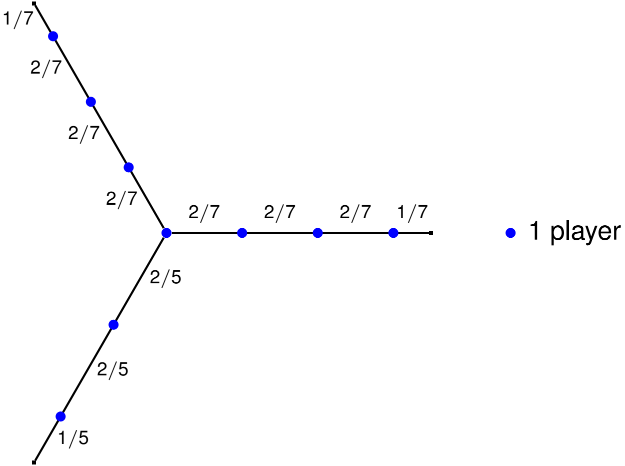


Figure: Good configuration on S_3 with 9 players.

Sketch of the existence proof

1. The graph $G = (X, E)$ and n are fixed. We want to construct a pure Nash equilibrium with n players on G. We fix a general parameter $\xi > 0$.

Sketch of the existence proof

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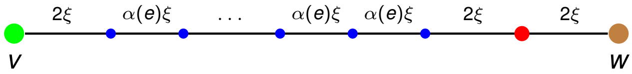
2. On each vertex we put a number of sellers that's equal to the degree of the vertex.

Sketch of the existence proof

3. On each edge we put a number of players $n(e)$ that only depends on the length $\lambda(e)$ of the edge and on ξ.



- 1 player
- 2 players
- $\deg(v)$ players
- $\deg(w)$ players

Where α is such that the number of players on e is $n(e)$.

Sketch of the existence proof

4. We prove that if ξ is small enough this profile of location is an equilibrium, with a number of player equal to

$$\sum_{e} n(e) = 3\,\mathrm{card}(E) + \sum_{e \in E} \left\lceil \frac{\lambda(e)}{2\xi} \right\rceil$$

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5. Can we find ξ such that $f(\xi) = n$?
6. No but we can find n' such that there exists ξ such that $f(\xi) = n'$, $n' \geq n$, and $n' - n \leq \mathrm{card}(E)$.
7. We select the equilibrium with n' players. We can remove up to one unnecessary player on each edge to have an equilibrium with n player.

Efficiency: Majorization

For a vector $\boldsymbol{z} = (z_1, \ldots, z_n)$, we denote $z_{[1]} \geq \cdots \geq z_{[n]}$ its decreasing rearrangement.

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Definition

Let $\boldsymbol{x}, \boldsymbol{y} \in [0, 1]^n$ be such

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$$

if, for all $k \in \{1, \ldots, n\}$

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}.$$

then we say that $\boldsymbol{x}$ is majorized by $\boldsymbol{y}$ ($\boldsymbol{x} \prec \boldsymbol{y}$).

Definition

A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is called Schur-convex if $\mathbf{x} \prec \mathbf{y}$ implies $\phi(\boldsymbol{x}) \leq \phi(\boldsymbol{y})$.

Proposition

If $\psi : \mathbb{R} \to \mathbb{R}$ is a convex function,

$$\phi(x_1, \ldots, x_n) = \sum_{i=1}^{n} \psi(x_i),$$

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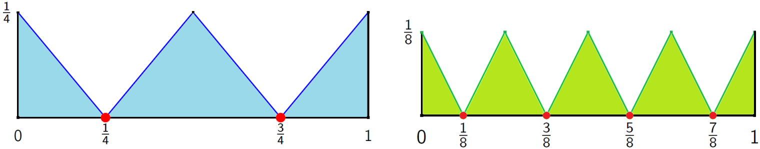

References I